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2002 J. Phys. A: Math. Gen. 35 6169

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Higher derivative fermionic field theories

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Received 26 March 2002, in final form 21 May 2002

Published 12 July 2002

Online at stacks.iop.org/JPhysA/35/6169

Abstract

We carry out the extension of the covariant Ostrogradski method to fermionic field theories. Higher derivative Lagrangians reduce to second-order differential ones with one explicit independent field for each degree of freedom.

PACS numbers: 04.62.+v, 11.10.–z

1. Introduction

Higher derivative (HD) field theories appear in many physical situations such as the Higgs model regularizations [1] or generalized electrodynamics [2–4], but their main interest lies in their use as gravitational actions. HD gravitational field models arise as effective low energy theories of the string [5] or are induced by quantum fields in a curved background [6]. Theories with quadratic curvature terms have been studied closely because they are renormalizable [7] in four dimensions. This property led to renormalization-group analyses in [8], which culminate in [9], including attempts to avoid the appearance of Weyl ghosts usually occurring in HD theories [10]. For these reasons, it is in the gravitational framework where essential progress has taken place. In particular, during the 1980s a mechanism was devised to deal with the Hamiltonian formulation of an arbitrary HD gauge field theory which was successfully applied to HD gravity [11]. Somewhat later, in the 1990s, a covariant differential order reduction for HD field theories was found [12]. These kinds of covariant techniques were used in the late 1990s to identify the propagating degrees of freedom (DOF) in both diff-invariant [13] and gauge-fixed HD gravity [14]. Moreover, some effort has been devoted to study general bosonic HD free theories [15, 16] as useful testbeds for HD gravity.

HD fermionic field theories have also been considered in the literature¹, for example, in the context of the effective action for the trace anomaly in conformal field theory [17], as a dynamical mechanism for fermionic mass generation and in the frame of Faddeev–Popov²

¹ Fermionic theories in which the field equations are differential equations of order greater than one are referred to as HD fermionic theories.

² Although the Faddeev–Popov compensating fields for HD gravity are not fermionic fields (their spins are 0 or 1) they are anticommuting fields. In this sense, they are different from the usual bosonic fields treated in [15, 16].

compensating Lagrangian for HD gravity [14]. Therefore, it would be very useful to generalize all the work done in the bosonic case to cover the presence of fermionic fields. This is one of the goals of the present paper; another is to provide a general framework for the differential order reduction methods developed for HD theories in [12] and treated in [13–15]. The starting point in the study of any field theory, including HD field theory, is the characterization of the propagating DOF. There are several ways to do this. The first, and standard, way follows a detailed analysis of the Ostrogradski phase space [15, 18] in order to characterize the reduced phase space [19], i.e. the subspace of the phase space where the physical DOF reside. However, when the theory is linear there exists a second and highly useful shortcut provided by the use of the covariant symplectic techniques of Witten and Crnković [20, 21]. The idea of these methods relies on the construction, directly through the action, of a symplectic form on the space of solutions to the field equations (covariant phase space) and uses it to generate conserved quantities (energy, angular momentum and so on) that characterize the propagating DOF. The covariant symplectic techniques have proved to be an essential tool in the classification of free theories and also in the identification of the propagating DOF [22]. We will see that they also allow us to complete, with *all* generality, the order reduction in HD theories.

When we are dealing with HD relativistic field theories, the story does not end once the DOF have been identified. The reason is that HD theories can be usually reinterpreted in terms of lower derivative (LD) theories that propagate (in the free limit) according to standard Lagrangians. The machinery necessary for this reinterpretation combines the use of the covariant Legendre transform [12] and a subsequent diagonalization in the fields [13]. This procedure has been developed in various examples: diff-invariant HD gravity [13], gauge fixed HD electromagnetism [4], gauge-fixed gravity [14] and HD scalars [15]. However, there are at least two issues left aside in these works. The first issue is the extension of the above results to fermions, covering the general derivative case³. The second and more interesting issue is to find a unifying point of view that encompasses all the previous works. The motivation of the present paper is to clarify these two points.

The paper is organized as follows. After this introduction, section 2 is devoted to reviewing the use of the covariant symplectic techniques for Dirac fields and to fix some conventions. Section 3 deals with a simple HD fermionic field theory in order to make it simpler to understand the general N -derivative fermionic theory—under the hypothesis of non-degenerate masses—treated in section 4. We end the paper with several comments and our conclusions (section 5). Some details of the computations and the general conventions are left to appendices A and B. Finally, appendix C indicates how the differential order reduction method proposed in section 4 can be extended to cover a wide class of HD field theories.

2. A review of the Dirac Lagrangian

Let us start with a quick review of the standard Dirac Lagrangian in order to fix conventions and notations. As is well known, the free propagation of spin 1/2 and mass m modes can be described through the Dirac Lagrangian \mathcal{L}_m^D , which can be written in two equivalent forms

$$\mathcal{L}_m^D = \frac{1}{2}[\bar{\psi}(i\overrightarrow{\not{\partial}} - m)\psi - \bar{\psi}(i\overleftarrow{\not{\partial}} + m)\psi] = \bar{\psi}(i\overrightarrow{\not{\partial}} - m)\psi - \partial_\mu \left(\frac{\bar{\psi}i\gamma^\mu\psi}{2} \right)$$

³ Within the frame of HD scalar theories and covariant Legendre transform, [15] fails to cover the diagonalization of the general derivative situation. The problem has been solved in [16] avoiding the covariant Legendre transform by using Lagrange multipliers and symplectic covariant techniques.

where m is a mass parameter, ψ is a Dirac spinor, $\not{\partial} := \gamma^\mu \partial_\mu$, γ^μ are the Dirac matrices, $\bar{\psi} := \psi^\dagger \gamma^0$, and \dagger denotes complex conjugation and transposition (see appendix A for a resumé of the conventions). The first and more symmetric expression is appropriate for analytical purposes and will be used in the following sections when we introduce the differential reduction order methods. The second is suitable to define the propagator. The Euler–Lagrange equations associated with the Dirac Lagrangian are

$$(i\not{\partial} - m)\psi = 0 \quad (1)$$

and the oriented $\bar{\psi}\text{--}\psi$ fermionic propagator is

$$\Delta_m^D := \frac{1}{i\not{\partial} - m} = -\frac{i\not{\partial} + m}{\square + m^2}.$$

The dynamics is governed by the field equations (1) that define the Dirac covariant phase space \mathcal{S}_m^D . To parametrize \mathcal{S}_m^D we solve the field equations using spatial Fourier transform and the notation introduced in appendix A. It is straightforward [23] to prove that $\psi \in \mathcal{S}_m^D$ if and only if

$$\psi(t, \vec{x}) = \sum_{\alpha=1,2} \int_{\mathbb{R}^3} \frac{d^3\vec{k}}{(2\pi)^3} \frac{m}{k^0} [a^\alpha(k)u^\alpha(k) e^{-ikx} + b^\alpha(k)v^\alpha(k) e^{ikx}] \quad (2)$$

where $u^\alpha(k)$, $v^\alpha(k)$ are basic spinors, $kx := k^0 t - \vec{k} \cdot \vec{x}$, $k^0 := \sqrt{\vec{k}^2 + m^2}$, and the functional parameters that distinguish one solution from another are encoded on the $a^\alpha(k)$ and $b^\alpha(k)$ fields.

Let us show how the symplectic covariant techniques [20] induce a symplectic form Ω_m^D in the space \mathcal{S}_m^D . First define the 2-form

$$\Omega^D = \int_{\Sigma} \omega_D^\mu d\sigma_\mu$$

where $d\sigma_\mu$ is the measure on a space-like hypersurface Σ ,

$$\omega_D^\mu := \mathbf{d} \frac{\partial \mathcal{L}_m^D}{\partial \psi_\mu} \wedge \mathbf{d}\psi - \mathbf{d}\bar{\psi} \wedge \mathbf{d} \frac{\partial \mathcal{L}_m^D}{\partial \bar{\psi}_\mu} = i \mathbf{d}\bar{\psi} \wedge \gamma^\mu \mathbf{d}\psi$$

and \mathbf{d} , \wedge are, respectively, the exterior derivative and the wedge product in the linear manifold \mathcal{S}_m^D .

The density ω_D^μ is divergence-free when restricted to \mathcal{S}_m^D , in other words $\partial_\mu \omega_D^\mu = 0$ modulo field equations. Then the restriction Ω_m^D of Ω^D on \mathcal{S}_m^D is a well-defined 2-form on this functional space, i.e. it is time-independent, and in parametrization (2) can be written as⁴

$$\Omega_m^D = i \sum_{\alpha=1,2} \int_{\mathbb{R}^3} \frac{d^3\vec{k}}{(2\pi)^3} \frac{m}{k^0} [\mathbf{d}a^{\alpha*}(\vec{k}) \wedge \mathbf{d}a^\alpha(\vec{k}) + \mathbf{d}b^\alpha(\vec{k}) \wedge \mathbf{d}b^{\alpha*}(\vec{k})]. \quad (3)$$

The symplectic form (3) determines the propagating DOF of the Dirac theory—the canonical pairs $a^{\alpha*}(\vec{k})\text{--}a^\alpha(\vec{k})$ and $b^{\alpha*}(\vec{k})\text{--}b^\alpha(\vec{k})$ — and can also be used to derive constants of motion if one considers the following version of the Noether theorem. When the symplectic form Ω is invariant under a group of transformations and we take a vector V tangent to an orbit of this group, it is straightforward to prove [21] that locally $i_V \Omega = \mathbf{d}H$, where $i_V \Omega$ denotes the contraction of V and Ω . The quantity H is the generator of the symmetry transformation corresponding to V . If the action is Poincaré invariant we obtain in this way the energy–momentum and the angular momentum densities (with the right symmetries in their tensor

⁴ z^* denotes the complex conjugate of z .

indices) by computing $i_V \Omega$ for vectors V describing translations and Lorentz transformations and writing the result as $\mathbf{d}H$. Noting that under a translation of parameter τ^μ we have $i_{V_\tau} \mathbf{d}a^\alpha(\vec{k}) = i\tau^\mu k_\mu a^\alpha(\vec{k})$ and $i_{V_\tau} \mathbf{d}b^\alpha(\vec{k}) = i\tau^\mu k_\mu b^\alpha(\vec{k})$, the energy of a solution is given by

$$H_m^D = \sum_{\alpha=1,2} \int_{\mathbb{R}^3} \frac{d^3\vec{k}}{(2\pi)^3} \frac{m}{k^0} k^0 [a^{\alpha*}(\vec{k})a^\alpha(\vec{k}) - b^\alpha(\vec{k})b^{\alpha*}(\vec{k})].$$

It is worthwhile pointing out that the minus sign between the a and the b fields forces us to take them as *anti-commuting* variables to ensure the positivity of the energy.

3. A simple HD fermionic theory

Once we feel comfortable with the notation, let us consider the following simple HD fermionic Lagrangian:

$$\mathcal{L}_{m_1 m_2}^{(2)} = \bar{\psi}(i\vec{\not{\partial}} - m_1)(i\vec{\not{\partial}} - m_2)\psi$$

with real mass parameters⁵ $m_1 < m_2$. It is straightforward to write down the HD field equations for this model,

$$(i\vec{\not{\partial}} - m_1)(i\vec{\not{\partial}} - m_2)\psi = 0 \quad (4)$$

whose space of solutions, $\mathcal{S}_{m_1 m_2}^{(2)}$, can be parametrized in terms of a sum of Dirac fields in the form

$$\psi = \psi_1 + \psi_2 \quad (5)$$

where $(i\vec{\not{\partial}} - m_l)\psi_l = 0$, for $l = 1, 2$. Explicitly, in the notation given in section 2,

$$\psi_l(t, \vec{x}) = \sum_{\alpha=1,2} \int_{\mathbb{R}^3} \frac{d^3\vec{k}}{(2\pi)^3} \frac{m_l}{k_l^0} [a_l^\alpha(k)u_l^\alpha(k) e^{-ik_l x} + b_l^\alpha(k)v_l^\alpha(k) e^{ik_l x}] \quad (6)$$

where, as in the previous section, $u_l^\alpha(k)$, $v_l^\alpha(k)$ are basic spinors defined in appendix A, $k_l x = k_l^0 t - \vec{k} \cdot \vec{x}$, $k_l^0 := \sqrt{\vec{k}^2 + m_l^2}$, and the parameters that distinguish one solution from another are encoded on the $a_l^\alpha(k)$ and $b_l^\alpha(k)$ fields. Also the HD propagator can be easily found as a sum of Dirac propagators

$$\Delta_{m_1 m_2}^{(2)} = \frac{1}{(i\vec{\not{\partial}} - m_1)(i\vec{\not{\partial}} - m_2)} = \frac{1}{m_2 - m_1} (\Delta_{m_2}^D - \Delta_{m_1}^D).$$

The physical interpretation is clear. The theory describes two LD fermionic DOF, a physical one (positive contribution to the energy) with mass m_2 and a Weyl ghost (negative contribution to the energy) with mass m_1 .

We can learn more about this theory by looking at it through the glass provided by the covariant symplectic techniques. The extension of the covariant symplectic techniques to the HD field theories was made in [24]. Following [24] the HD Lagrangian has an associated 2-form

$$\Omega^{(2)} = \int_{\Sigma} \omega_{(2)}^\mu d\sigma_\mu$$

where

$$\begin{aligned} \omega_{(2)}^\mu &:= \mathbf{d} \left(\frac{\partial \mathcal{L}_{m_1 m_2}^{(2)}}{\partial \psi_\mu} - \partial_\nu \frac{\partial \mathcal{L}_{m_1 m_2}^{(2)}}{\partial \psi_{\mu\nu}} \right) \wedge \mathbf{d}\psi + \mathbf{d} \frac{\partial \mathcal{L}_{m_1 m_2}^{(2)}}{\partial \psi_{\mu\nu}} \wedge \mathbf{d}\psi_\nu \\ &= -i(m_1 + m_2) \mathbf{d}\bar{\psi} \wedge \gamma^\mu \mathbf{d}\psi + (\partial^\mu \mathbf{d}\bar{\psi}) \wedge \mathbf{d}\psi - \mathbf{d}\bar{\psi} \wedge \partial^\mu \mathbf{d}\psi. \end{aligned}$$

⁵ The problems associated with the presence of degenerate and/or complex ‘masses’ are detailed in [16].

It is easy to show that $\omega_{(2)}^\mu$ is real and, modulo field equations, $\partial_\mu \omega_{(2)}^\mu = 0$. Then we can compute the symplectic form, $\Omega_{m_1 m_2}^{(2)}$, over the space of propagating DOF, $\mathcal{S}_{m_1 m_2}^{(2)}$, making use of the parametrization given by equations (5)–(6). This leads to a well-defined 2-form on $\mathcal{S}_{m_1 m_2}^{(2)}$, namely

$$\Omega_{m_1 m_2}^{(2)} = (m_2 - m_1) (\Omega_{m_2}^D - \Omega_{m_1}^D)$$

and Noether's theorem gives us the energy

$$H_{m_1 m_2}^{(2)} = (m_2 - m_1) (H_{m_2}^D - H_{m_1}^D).$$

This supports the intuitive interpretation of $\mathcal{L}_{m_1 m_2}^{(2)}$ given by the propagator decomposition as a theory describing two Dirac fields. In fact, the symplectic form confirms that this is the right interpretation and also, through the Noether theorem, that the field ψ_1 is a Weyl ghost. It contributes to the energy with a wrong sign, destroying the semi-boundedness of the Hamiltonian and consequently the unitarity of the quantum formulation of the theory.

Finally, let us show how it is possible to find a covariant Legendre transform connecting the HD theory with an LD theory where the usual Dirac fields are explicit. To this end, we define the first-order differential operator $D := i\rlap{/}\partial$ that satisfies $\int_{\mathbb{R}^4} \bar{\psi}_1 (D\psi_2) = \int_{\mathbb{R}^4} \overline{(D\psi_1)} \psi_2$, and rewrite the Lagrangian in terms of this object, namely, modulo total derivatives,

$$\mathcal{L}_{m_1 m_2}^{(2)}(\psi, \bar{\psi}, D\psi, \overline{D\psi}) := \overline{D\psi} D\psi - \frac{m_1 + m_2}{2} (\overline{D\psi} \psi + \bar{\psi} D\psi) + m_1 m_2 \bar{\psi} \psi.$$

Then we introduce a generalized Legendre transform [12] with respect to the momenta generated by the D -operator⁶,

$$\pi := \frac{\partial \mathcal{L}_{m_1 m_2}^{(2)}}{\partial \overline{D\psi}} = D\psi - \frac{m_1 + m_2}{2} \psi \quad \bar{\pi} := \frac{\partial \mathcal{L}_{m_1 m_2}^{(2)}}{\partial D\psi} = \overline{D\psi} - \frac{m_1 + m_2}{2} \bar{\psi}. \quad (7)$$

This transformation is not singular and it permits the inversion of the derivative of ψ as a function $D\psi = v(\pi, \psi)$ given by

$$v(\pi, \psi) := \pi + \frac{m_1 + m_2}{2} \psi \quad \overline{v(\pi, \psi)} := \bar{\pi} + \frac{m_1 + m_2}{2} \bar{\psi}.$$

The Legendre transform (7) has an associated Hamiltonian

$$\begin{aligned} \mathcal{H}_{m_1 m_2}^{(2)}(\psi, \bar{\psi}, \pi, \bar{\pi}) &:= \bar{\pi} v(\pi, \psi) + \overline{v(\pi, \psi)} \pi - \mathcal{L}_{m_1 m_2}^{(2)}(\psi, \bar{\psi}, v(\pi, \psi), \overline{v(\pi, \psi)}) \\ &= \bar{\pi} \pi + \frac{m_1 + m_2}{2} (\bar{\psi} \pi + \bar{\pi} \psi) + \left(\frac{m_1 - m_2}{2} \right)^2 \bar{\psi} \psi \end{aligned}$$

and the HD dynamics (4) can be described now by means of the Euler–Lagrange equations derived from the Helmholtz LD Lagrangian

$$\mathcal{L}_H^{(2)}(\psi, \bar{\psi}, \pi, \bar{\pi}; D\psi, \overline{D\psi}) := \bar{\pi} D\psi + \overline{D\psi} \pi - \mathcal{H}_{m_1 m_2}^{(2)}(\psi, \bar{\psi}, \pi, \bar{\pi}).$$

Although $\mathcal{L}_H^{(2)}$ is an LD Lagrangian classically equivalent to $\mathcal{L}_{m_1 m_2}^{(2)}$, it does not explicitly exhibit the propagating DOF. However, the diagonalization can be carried out taking into account the special structure of the covariant phase space. Defining the new fields ψ_1 and ψ_2 in the form

$$\psi =: \psi_1 + \psi_2 \quad (8)$$

$$\pi =: \frac{m_1 - m_2}{2} (\psi_1 + \psi_2) \quad (9)$$

⁶ We must treat a field and its conjugate as independent variables in the Legendre transform in order to allow the presence of Grassmannian fields [14]. Then, the derivatives with respect to $D\psi$ and $\overline{D\psi}$ are respectively left and right derivatives.

the Helmholtz Lagrangian decouples the propagating modes,

$$\mathcal{L}_H^{(2)} = (m_2 - m_1) (\mathcal{L}_{m_2}^D - \mathcal{L}_{m_1}^D).$$

The relative minus sign shows the presence of a Weyl ghost.

The diagonalization (8)–(9) can be found by means of a very natural reasoning. Over the solution space, where the DOF reside, we can write $\psi = \psi_1 + \psi_2$ with $(i\partial - m_l)\psi_l = 0$. Then, on this space

$$\psi = \psi_1 + \psi_2 \quad \pi = D\psi - \frac{m_1 + m_2}{2}\psi = \frac{m_1 - m_2}{2}(\psi_1 - \psi_2).$$

The above relations, which coincide with the proposed diagonalization, indicate how the DOF are encoded within the fields ψ and π . Moreover, the relation $\{\psi, \pi\} \leftrightarrow \{\psi_1, \psi_2\}$ is invertible because of the invertibility of the Legendre transform (7).

We are ready to generalize the above results, obtained in a very simple 2-derivative framework, to the general N -derivative situation.

4. N -derivative fermionic theory

In this section, we consider the general N -derivative Lagrangian

$$\mathcal{L}_m^{(N)} := \overline{\psi} \prod_{l=1}^N (i\overleftrightarrow{\partial} - m_l)\psi$$

where $(i\partial - m_l)$ are Dirac operators with real mass parameters m_l that have been ordered following the l -index ordering, i.e. $m_l < m_{l'}$ when $l < l'$.

The propagating DOF described by $\mathcal{L}_m^{(N)}$ can be read out from the algebraic decomposition for the HD propagator $\Delta_m^{(N)}$ in terms of Dirac propagators,

$$\Delta_m^{(N)} := \frac{1}{\prod_{l=1}^N (i\partial - m_l)} = \sum_{l=1}^N \frac{\Delta_{m_l}^D}{\prod_{l' \neq l} (m_l - m_{l'})}.$$

Note that the sign alternates in the coefficients $\prod_{l' \neq l} (m_l - m_{l'})$, so the occurrence of Weyl ghosts is expected. Mathematically, the propagating DOF are points in the space of solutions to the field equations that we refer to in the following as $\mathcal{S}_m^{(N)}$. The covariant phase space $\mathcal{S}_m^{(N)}$ is defined by the Euler–Lagrange equations,

$$\prod_{l=1}^N (i\partial - m_l)\psi = 0.$$

These equations can be solved by means of a linear combination of Dirac fields in the form

$$\psi = \sum_{l=1}^N \psi_l$$

where

$$(i\partial - m_l)\psi_l = 0. \quad (10)$$

As in previous sections, the standard parametrization for the ψ_l fields satisfying equation (10), which provides us with an explicit parametrization of $\mathcal{S}_m^{(N)}$, is

$$\psi_l(t, \vec{x}) = \sum_{\alpha=1,2} \int_{\mathbb{R}^3} \frac{d^3\vec{k}}{(2\pi)^3} \frac{m_l}{k_l^0} [a_l^\alpha(k) u_l^\alpha(k) e^{-ik_l x} + b_l^\alpha(k) v_l^\alpha(k) e^{ik_l x}]. \quad (11)$$

The decomposition of the HD propagator as a sum of Dirac propagators, with alternating sign coefficients, and the decomposition of the space of solutions as a direct sum of Dirac spaces indicate that the theory represents the propagation of N -Dirac fields, some of them physical and some of them Weyl ghosts. This is the case, and it is possible to give a more precise proof of this fact. The Lagrangian $\mathcal{L}_m^{(N)}$ induces a symplectic form in the space of spinor fields by means of

$$\Omega^{(N)} = \int_{\Sigma} \omega_{(N)}^{\mu} d\sigma_{\mu}$$

where

$$\begin{aligned} \omega_{(N)}^{\mu} := & \mathbf{d} \left(\frac{\partial \mathcal{L}_m^{(N)}}{\partial \psi_{\mu}} - \partial_{\mu_1} \frac{\partial \mathcal{L}_m^{(N)}}{\partial \psi_{\mu\mu_1}} + \dots + (-1)^{N-1} \partial_{\mu_1} \dots \partial_{\mu_{N-1}} \frac{\partial \mathcal{L}_m^{(N)}}{\partial \psi_{\mu\mu_1 \dots \mu_{N-1}}} \right) \wedge \mathbf{d}\psi \\ & + \mathbf{d} \left(\frac{\partial \mathcal{L}_m^{(N)}}{\partial \psi_{\mu\nu_1}} - \partial_{\mu_1} \frac{\partial \mathcal{L}_m^{(N)}}{\partial \psi_{\mu\nu_1\mu_1}} + \dots + (-1)^{N-2} \partial_{\mu_1} \dots \partial_{\mu_{N-2}} \frac{\partial \mathcal{L}_m^{(N)}}{\partial \psi_{\mu\nu_1\mu_1 \dots \mu_{N-2}}} \right) \wedge \mathbf{d}\psi_{\nu_1} \\ & + \dots + \mathbf{d} \frac{\partial \mathcal{L}_m^{(N)}}{\partial \psi_{\mu\nu_1 \dots \nu_{N-1}}} \wedge \mathbf{d}\psi_{\nu_1 \dots \nu_{N-1}}. \end{aligned} \quad (12)$$

This is so because over the $\mathcal{S}_m^{(N)}$ space the 2-form density $\omega_{(N)}^{\mu}$ satisfies $\partial_{\mu} \omega_{(N)}^{\mu} = 0$. Hence, the restriction of $\Omega^{(N)}$ to $\mathcal{S}_m^{(N)}$ —that we refer to as $\Omega_m^{(N)}$ —is a well-defined 2-form on this functional space, i.e. it is time-independent. In fact, it is straightforward, but highly tedious (see appendix B), to compute this restriction in the parametrization given by (10) to obtain

$$\Omega_m^{(N)} = \sum_{l=1}^N \frac{\Omega_{m_l}^D}{\prod_{l \neq l'}^N (m_l - m_{l'})}. \quad (13)$$

Consequently, the DOF of the theory are a sum of Dirac DOF. Finally the energy, computed through the Noether theorem, takes the form

$$H_m^{(N)} = \sum_{l=1}^N \frac{H_{m_l}^D}{\prod_{l \neq l'}^n (m_l - m_{l'})}.$$

Because the sign alternates in the coefficients $\prod_{l \neq l'}^N (m_l - m_{l'})$, namely $\text{sg}(\prod_{l \neq l'}^N (m_l - m_{l'})) = (-1)^{N+l}$, some of the fields are physical (positive contribution to the energy) and some are Weyl ghosts (negative contribution to the energy).

Once we have identified the propagating DOF—and due to the special form of the propagator, the covariant phase space, the symplectic form and energy that can be obtained as linear combinations of Dirac objects—it is plausible to presume the existence of a mapping that transforms the original HD theory into a sum of LD Dirac theories. To this end, as we did in section 3, it is convenient to follow a series of preliminary steps. First, define the differential operator $D := i\partial$ and then expand the differential kernel $\prod_{l=1}^N (D - m_l)$ appearing in $\mathcal{L}_m^{(N)}$ in the form

$$\prod_{l=1}^N (D - m_l) \equiv \sum_{l=0}^N c_l D^{N-l}$$

where

$$c_0 := 1 \quad c_l := (-1)^l \sum_{a_1 < \dots < a_l} m_{a_1} \dots m_{a_l} \quad l = 1, \dots, N.$$

A last technical remark is still in order. The heavy algebra involved in the following forces us to treat the N odd and N even cases separately. We work out in detail the $N = 2n$ case

and refer to the bosonic framework considered in [15, 25] to understand certain peculiarities present for $N = 2n - 1$.⁷

Modulo total derivatives, the HD Lagrangian $\mathcal{L}_m^{(2n)}$ can be rewritten in a more convenient form

$$\mathcal{L}^{(2n)} = \sum_{l=0}^n c_{2l} \overline{D^{n-l}\psi} D^{n-l}\psi + \sum_{l=1}^n \frac{c_{2l-1}}{2} [D^{n-l}\psi D^{n+1-l}\psi + \overline{D^{n+1-l}\psi} D^{n-l}\psi].$$

This expression suggests the introduction of the Ostrogradski-like variables

$$\chi_r := D^{r-1}\psi \quad \overline{\chi}_r := \overline{D^{r-1}\psi} \quad r = 1, \dots, n \quad (14)$$

and their corresponding momenta by means of a covariant Legendre transform

$$\begin{aligned} \pi_n &:= \frac{\partial \mathcal{L}}{\partial \overline{D^n \psi}} = D\chi_n + \frac{c_1}{2}\chi_n & \overline{\pi}_n &:= \frac{\partial \mathcal{L}}{\partial D^n \psi} = \overline{D}\chi_n + \frac{c_1}{2}\overline{\chi}_n. \\ \pi_r &:= \frac{\partial \mathcal{L}}{\partial \overline{D^r \psi}} + D\pi_{r+1} & \overline{\pi}_r &:= \frac{\partial \mathcal{L}}{\partial D^r \psi} + \overline{D}\overline{\pi}_{r+1} \quad r = 1, \dots, n-1. \end{aligned} \quad (15)$$

It is easy to show that this Legendre transform is non-singular due to the invertibility of the highest derivatives $D\chi_n$ and $\overline{D}\chi_n$ in terms of the variables χ and π , namely

$$v_n(\chi, \pi) := D\chi_n = \pi_n - \frac{c_1}{2}\chi_n \quad \overline{v}_n(\chi, \pi) := \overline{D}\chi_n = \overline{\pi}_n - \frac{c_1}{2}\overline{\chi}_n.$$

Thus it is possible to find an expression for the Lagrangian in terms of these new variables,

$$\mathcal{L}_m^{(2n)} = \overline{\pi}_n \pi_n - \frac{c_1}{2} \overline{\chi}_n \chi_n + \sum_{r=0}^{n-1} c_{2r} \overline{\chi}_{n-r} \chi_{n-r} + \sum_{r=0}^{n-2} \frac{c_{2r-1}}{2} [\overline{\chi}_{n-r} \chi_{n-1-r} + \overline{\chi}_{n-1-r} \chi_{n-r}].$$

The Ostrogradski-like Hamiltonian associated with the Legendre transform is

$$\begin{aligned} \mathcal{H}_m^{(2n)} &:= \overline{\pi}_n v_n(\chi, \pi) + \overline{v}_n(\chi, \pi) \pi_n + \sum_{r=1}^{n-1} \overline{\pi}_r \chi_{r+1} \\ &\quad + \sum_{r=1}^{n-1} \overline{\chi}_{r+1} \pi_r - \mathcal{L}_m^{(N)}(\chi, \overline{\chi}; v_n(\chi, \pi), \overline{v}_n(\chi, \pi)) \\ &= \overline{\pi}_n \pi_n - \frac{c_1}{2} [\overline{\pi}_n \chi_n + \overline{\chi}_n \pi_n] + \frac{c_1^2}{2} \overline{\chi}_n \chi_n + \sum_{r=1}^{n-1} [\overline{\pi}_r \chi_{r+1} + \overline{\chi}_{r+1} \pi_r] \\ &\quad - \sum_{r=0}^{n-1} c_{2r} \overline{\chi}_{n-r} \chi_{n-r} - \sum_{r=0}^{n-2} \frac{c_{2r-1}}{2} [\overline{\chi}_{n-r} \chi_{n-1-r} + \overline{\chi}_{n-1-r} \chi_{n-r}]. \end{aligned}$$

Finally, the Helmholtz Lagrangian—the LD Lagrangian, classically equivalent to $\mathcal{L}_m^{(2n)}$, generated by the Legendre transform—is

$$\mathcal{L}_H^{(2n)} := \sum_{r=1}^n [\overline{\pi}_r D\chi_r + \overline{D}\chi_r \pi_r] - \mathcal{H}_m^{(2n)}.$$

As usual in the covariant Legendre procedure [15, 16], the expression for $\mathcal{L}_H^{(2n)}$ does not exhibit the propagating DOF. However, it is straightforward to find a new set of variables in terms of

⁷ In the N odd case, the definition of the highest momentum yields a constraint—that can be easily taken into account—while the field derivative is worked out from the next momentum definition. Once the constraint is solved, the odd case is exactly analogue to the even case.

which the propagation is explicit. Specifically, it suffices to introduce the new set of ψ_l -fields⁸ through

$$\chi_r := \sum_{l=1}^{2n} m_l^{r-1} \psi_l \quad (16)$$

$$\bar{\chi}_r := \sum_{l=1}^{2n} m_l^{r-1} \bar{\psi}_l \quad (17)$$

$$\pi_r := \sum_{l=1}^{2n} \left(\sum_{k=0}^{2(n-r)} c_k m_l^{2n-r-k} + \frac{c_{2(n-r)+1}}{2} m_l^{r-1} \right) \psi_l \quad (18)$$

$$\bar{\pi}_r := \sum_{l=1}^{2n} \left(\sum_{k=0}^{2(n-r)} c_k m_l^{2n-r-k} + \frac{c_{2(n-r)+1}}{2} m_l^{r-1} \right) \bar{\psi}_l. \quad (19)$$

In these variables, and modulo total derivatives, the Helmholtz Lagrangian becomes diagonal,

$$\mathcal{L}_H^{(2n)} = \sum_{l=1}^{2n} \prod_{l' \neq l}^{2n} (m_l - m_{l'}) \mathcal{L}_{m_l}^D.$$

The key observation to find the diagonalization (16)–(19) relies on the following simple fact. The Ostrogradski variables χ_r and momenta π_r can be expressed by means of a one-to-one map in terms of the original HD variable ψ and its derivatives $D^k \psi$, namely

$$\chi_r = D^{r-1} \psi \quad (20)$$

$$\pi_r = \sum_{k=0}^{2(n-r)} c_k D^{2n-r-k} \psi + \frac{c_{2(n-r)+1}}{2} D^{r-1} \psi. \quad (21)$$

When we restrict relations (20)–(21) to the space of solutions parametrized by (6)—that is, over the propagating DOF space—we re-obtain the one-to-one linear relations (16)–(19) between the variables $\{\chi_r, \pi_r\}$ and $\{\psi_l\}$. This is the reason why this linear redefinition diagonalizes \mathcal{L}_H^{2n} .

All the preceding results can be generalized to cover a wide class of linear theories including those considered in previous works [4, 16]. In particular, the procedure followed here resolves the deficiencies inherent to more primitive approaches [15]. Specifically, it leads us to the diagonalization for the N -derivative theory within the framework of the covariant Legendre mapping. We summarize in appendix C the essential steps and requirements that allow us, following the lines of this section, to transform an HD linear theory into an LD linear theory where the propagating DOF are explicit.

5. Conclusions and comments

We have proved the equivalence of HD fermionic field theories and an LD counterpart where the DOF are explicit by means of a covariant Legendre transform [12] and a subsequent diagonalization. The previous attempts to solve this kind of problem, which considered only scalar field theories [15], failed to find the general diagonalization to an arbitrary differential order due to the heavy algebra involved. A way out of this was given in [16], in a more general framework than [15]—and also dealing with bosonic field theories—but the solution proposed

⁸ At this point, the ψ_l fields are simply a new set of variables. We follow the same notation used to denote the Dirac fields in (11). We will show that these new fields diagonalize the Helmholtz Lagrangian in terms of the Dirac fields.

there abandons the use of the Legendre transform in favour of the Lagrange multipliers method. The use of the covariant symplectic techniques in combination with the covariant Legendre transform avoids the algebraic problems of [15] permitting us to find an explicit formula for the DOF diagonalization at every differential order and to generalize the previous results to the fermionic case.

The approach that we follow here is exportable to more general theories—classical mechanics, HD gravity and so on—as we schematically show in appendix C. It is also important to realize that the presence of gauge symmetries in the theory does not change the results in any way. When dealing with HD theories written in terms of differential forms, such as those considered in [16], the role of the D -operator ($D = i\rlap{/}\partial$ in the present work) can be simulated by an operator constructed by means of the exterior differential d and its dual δ , for example $D = \delta d$. Even considering the possible existence of gauge symmetries ($D d\Lambda = 0$, for any field Λ) the analogue of the Legendre mapping (14)–(15) is still non-singular and therefore the reduction order procedure works exactly as in section 4.

Another remarkable fact is that, in spite of the fact that it is always possible to reduce the differential order by means of a covariant Legendre mapping, the diagonalization of the resulting LD theory in terms of a sum of standard theories is not always possible [16] (but when possible, the method presented here allows us to find it). To fix ideas, let us return to the fermionic case where the more general HD, Lorentz invariant, Lagrangian for a spinor ψ without internal indices takes the form

$$\mathcal{L}^{(n)} = \bar{\psi} (D^n + c_1 D^{n-1} + \dots + c_{n-1} D + c_n) \psi$$

where c_i are real parameters with the appropriate dimensions. As is well known, the polynomial $D^n + c_1 D^{n-1} + \dots + c_{n-1} D + c_n$ can be factorized in terms of its roots. In section 4 we have made the assumption that all the roots are different⁹. This is a necessary hypothesis in order to succeed in the diagonalization process. However, if we consider an HD ‘mass-degenerate’ model, the diagonalization cannot be carried out. For example, starting with

$$\mathcal{L}_{mm}^{(2)} = \bar{\psi} (D - m)^2 \psi$$

the Legendre procedure presented in section 4 leads us to the LD Hemholtz Lagrangian,

$$\mathcal{L}_H^{(2)} = (\bar{\psi} \ \bar{\pi}) \begin{pmatrix} 0 & D - m \\ D - m & -1 \end{pmatrix} \begin{pmatrix} \psi \\ \pi \end{pmatrix}.$$

The propagator associated with $\mathcal{L}_H^{(2)}$ is

$$\Delta_{(\psi\pi)} = \frac{1}{(D - m)^2} \begin{pmatrix} 1 & D - m \\ D - m & 0 \end{pmatrix}.$$

As is clear, this propagator cannot be diagonalized by means of a linear redefinition on the fields (ψ, π) not involving differential operators. This is so because $1/(D - m)^2$ is not a linear combination of Dirac propagators, and hence it is not possible to find a c -number matrix Q such that $Q^\dagger \Delta_{(\psi\pi)} Q$ becomes diagonal with Dirac propagators on its diagonal elements. The same conclusion can be reached with an elementary analysis of the covariant phase space.

Finally, we want to point out that the diagonalization process of the LD equivalent theory, obtained by Legendre transform, strongly relies on the decomposition of the HD solution space as a direct sum of LD ones. This decomposition is tied to the decomposition of the HD propagator as a sum of simpler pieces or, at least, on its relationships with simpler propagators. Then, in order to make practical use of the differential order reductions presented in the paper,

⁹ We have also assumed their positivity. This is a necessary requirement if we want these parameters to be physical masses, but it can be relaxed for diagonalization purposes.

knowledge of the properties of the Green functions of HD differential operators is needed. Some results in this direction can be found in [26] where, by means of heat kernel methods, the Green functions of a wide class of second-order differential operators have been studied and their relationship with LD counterparts is pointed out. These results are a key ingredient if one wants to find the LD-diagonalized equivalent of such an HD theory by means of a covariant Legendre transform.

Acknowledgments

The author wishes to thank F Barbero, J Julve and F J de Urries for several interesting discussions. This work was supported by a Spanish Ministry of Education and Culture fellowship co-financed by the European Social Fund.

Appendix A. Conventions

We use the Minkowski metric $\eta_{\mu\nu} = \text{diag}(+1, -1, -1, -1)$ to lower and raise spacetime indices and use the Einstein convention for summation over repeated indices. The derivatives with respect to coordinates are sometimes abbreviated as

$$\partial_{\mu_1 \dots \mu_k} := \frac{\partial^k}{\partial_{\mu_1} \dots \partial_{\mu_k}}$$

and $\square := \partial_\mu \partial^\mu = \partial_0^2 - \vec{\partial}^2$ is the d'Alembertian operator.

For any $X^{\mu_1 \dots \mu_k}$, $X^{\{\mu_1 \dots \mu_k\}}$ represents its index symmetrization, namely

$$X^{\{\mu_1 \dots \mu_k\}} := \frac{1}{k!} \sum_{\pi \in \Pi^{(k)}} X^{\mu_{\pi_1} \dots \mu_{\pi_k}}$$

where $\Pi^{(k)}$ denotes the permutation group of k -elements.

The Dirac γ matrices satisfy

$$\gamma^{\{\mu_1 \gamma^{\mu_2\}} = \eta^{\mu_1 \mu_2}$$

with γ^0 Hermitian and γ^i anti-Hermitian. As usual $\not{k} := k_\mu \gamma^\mu$.

Dirac spinors $u^\alpha(k)$ and $v^\alpha(k)$ are a basis of solutions to the Dirac equations,

$$(\not{k} - m) u^\alpha(k) = 0 \quad (\not{k} + m) v^\alpha(k) = 0.$$

They depend implicitly on the mass m and explicitly on the on-shell momentum k with $k^0 := \sqrt{\vec{k}^2 + m^2}$, and the α index that labels the polarizations. On this basis, and by making use of the spatial Fourier transform, any spinor field can be expressed as

$$\psi(t, \vec{x}) = \sum_{\alpha=1,2} \int_{\mathbb{R}^3} \frac{d^3 \vec{k}}{(2\pi)^3} \frac{m}{k^0} [a^\alpha(k, t) u^\alpha(k) e^{i\vec{k} \cdot \vec{x}} + b^\alpha(k, t) v^\alpha(k) e^{-i\vec{k} \cdot \vec{x}}].$$

The functions $a^\alpha(k, t)$ and $b^\alpha(k, t)$ take the special form $a^\alpha(k, t) = a^\alpha(k) e^{-ik^0 t}$ and $b^\alpha(k, t) = b^\alpha(k) e^{ik^0 t}$ when considering spinor fields that satisfy the Dirac equation $(i\not{\partial} - m)\psi = 0$.

The basic spinors u and v satisfy the following normalization properties:

$$\bar{u}^\alpha(k) u^\beta(k) = \delta_{\alpha\beta} \quad \bar{v}^\alpha(k) v^\beta(k) = -\delta_{\alpha\beta} \quad \bar{v}^\alpha(k) u^\beta(k) = \bar{u}^\alpha(k) v^\beta(k) = 0.$$

$$u^{\alpha\dagger}(k) u^\beta(k) = \delta_{\alpha\beta} \frac{\sqrt{\vec{k}^2 + m^2}}{m} \quad v^{\alpha\dagger}(k) v^\beta(k) = \delta_{\alpha\beta} \frac{\sqrt{\vec{k}^2 + m^2}}{m}.$$

$$v^{\alpha\dagger}(k^0, -\vec{k}) u^\beta(k^0, \vec{k}) = u^{\alpha\dagger}(k^0, -\vec{k}) v^\beta(k^0, \vec{k}) = 0$$

where the conjugate spinors \bar{u} are defined as $\bar{u} := u^\dagger \gamma^0$, and \dagger denotes transposition and complex conjugation.

Appendix B. Some remarks on the computation of the symplectic form

We summarize here the main steps involved in the calculation of the symplectic form (13) of section 4. The symplectic form is given [24] through space integration of the density

$$\begin{aligned} \omega_{(N)}^\mu &:= \mathbf{d} \left(\frac{\partial \mathcal{L}_m^{(N)}}{\partial \psi_\mu} - \partial_{\mu_1} \frac{\partial \mathcal{L}_m^{(N)}}{\partial \psi_{\mu\mu_1}} + \dots + (-1)^{N-1} \partial_{\mu_1} \dots \partial_{\mu_{N-1}} \frac{\partial \mathcal{L}_m^{(N)}}{\partial \psi_{\mu\mu_1 \dots \mu_{N-1}}} \right) \wedge \mathbf{d}\psi \\ &+ \mathbf{d} \left(\frac{\partial \mathcal{L}_m^{(N)}}{\partial \psi_{\mu\nu_1}} - \partial_{\mu_1} \frac{\partial \mathcal{L}_m^{(N)}}{\partial \psi_{\mu\nu_1\mu_1}} + \dots + (-1)^{N-2} \partial_{\mu_1} \dots \partial_{\mu_{N-2}} \frac{\partial \mathcal{L}_m^{(N)}}{\partial \psi_{\mu\nu_1\mu_1 \dots \mu_{N-2}}} \right) \wedge \mathbf{d}\psi_{\nu_1} \\ &+ \dots + \mathbf{d} \frac{\partial \mathcal{L}_m^{(N)}}{\partial \psi_{\mu\nu_1 \dots \nu_{N-1}}} \wedge \mathbf{d}\psi_{\nu_1 \dots \nu_{N-1}}. \end{aligned}$$

In order to compute the derivatives of the Lagrangian that appear in the definition of $\omega_{(N)}^\mu$ it is convenient to rewrite

$$\mathcal{L}_m^{(N)}(\psi, \psi_{\mu_1}, \dots, \psi_{\mu_1 \dots \mu_N}) := \bar{\psi} \prod_{l=1}^N (i\partial - m_l) \psi = \sum_{k=0}^N c_{N-k} i^k \bar{\psi} \partial^k \psi$$

where the constants c_k are defined in terms of mass products

$$c_0 := 1 \quad c_k := (-1)^k \sum_{a_1 < \dots < a_k} m_{a_1} \dots m_{a_k}.$$

Thus, in this notation,

$$\frac{\partial \mathcal{L}_m^{(N)}}{\partial \psi_{\mu_1 \dots \mu_k}} = c_{N-k} i^k \bar{\psi} \gamma^{\{\mu_1 \dots \mu_k\}} \tag{B1}$$

where the symmetrization of the Dirac gamma products can be expressed in terms of the inverse of the Minkowskian metric $\eta^{\mu\nu}$ in one of the following forms:

$$\begin{aligned} \gamma^{\{\mu_1 \dots \mu_{2k}\}} &= \eta^{\{\mu_1 \mu_2 \dots \mu_{2k-1} \mu_{2k}\}} \quad (\text{even}) \\ \gamma^{\{\mu_1 \dots \mu_{2k-1}\}} &= \gamma^{\{\mu_1 \eta^{\mu_2 \mu_3} \dots \eta^{\mu_{2k-2} \mu_{2k-1}}\}} \quad (\text{odd}) \quad k \in \mathbb{N}. \end{aligned} \tag{B2}$$

Plugging (B2) into (B1) and taking care of the combinatorics, we found that the terms of the density $\omega_{(N)}^\mu$ belong to one of the following four categories:

- (1) $\partial_{\mu_1 \dots \mu_{2k-1}} \frac{\partial \mathcal{L}_m^{(N)}}{\partial \psi_{\mu\mu_1 \dots \mu_{2k-1}}} = (-1)^k c_{N-2k} \square^{k-1} \partial^\mu \bar{\psi}.$
- (2) $\partial_{\mu_1 \dots \mu_{2k}} \frac{\partial \mathcal{L}_m^{(N)}}{\partial \psi_{\mu\mu_1 \dots \mu_{2k}}} = \frac{(-1)^k i c_{N-2k-1}}{2k+1} (\square^k \partial^\mu \bar{\psi} \gamma^\mu + 2k \square^{k-1} \partial^\mu \partial_\sigma \bar{\psi} \gamma^\sigma).$
- (3) $\partial_{\mu_1 \dots \mu_{2k}} \frac{\partial \mathcal{L}_m^{(N)}}{\partial \psi_{\mu\nu\mu_1 \dots \mu_{2k}}} = \frac{(-1)^{k+1} c_{N-2k-2}}{2k+1} (\square^k \eta^{\mu\nu} \bar{\psi} + 2k \square^{k-1} \partial^\mu \partial^\nu \bar{\psi}).$
- (4) $\partial_{\mu_1 \dots \mu_{2k-1}} \frac{\partial \mathcal{L}_m^{(N)}}{\partial \psi_{\mu\nu\mu_1 \dots \mu_{2k-1}}} = \frac{(-1)^k i c_{N-2k-1}}{2k+1} (\partial^\mu \square^{k-1} \bar{\psi} \gamma^\nu + \partial^\nu \square^{k-1} \bar{\psi} \gamma^\mu + \eta^{\mu\nu} \partial_\sigma \square^{k-1} \bar{\psi} \gamma^\sigma + 2(k-1) \partial^\mu \partial^\nu \partial_\sigma \square^{k-2} \bar{\psi} \gamma^\sigma).$

Now making use of parametrization (11), that is $\psi = \sum_{l=0}^N \psi_l$, where

$$\psi_l(t, \vec{x}) = \sum_{\alpha=1,2} \int_{\mathbb{R}^3} \frac{d^3 \vec{k}}{(2\pi)^3} \frac{m_l}{k_l^0} [a_l^\alpha(k) u_l^\alpha(k) e^{-ik_l x} + b_l^\alpha(k) v_l^\alpha(k) e^{ik_l x}] \quad k_l^0 = \sqrt{\vec{k}^2 + m^2}$$

remembering that the constants c_k are defined in terms of the masses m_l and using the properties of the spinors u_l and v_l detailed in appendix A it is straightforward, but slightly tedious, to derive equation (13) for the symplectic form.

Appendix C. The underlying idea

The procedure developed in the main body of the paper for spinorial fields can be straightforwardly extended to cover all linear models for which the field equations can be derived from a variational principle of the form

$$S_{HD}[\phi] = \langle \phi | (D + M_1) \cdots (D + M_N) \phi \rangle \quad (C1)$$

where M_i are ‘mass’ parameters (usually $M_i = -m_i$ when dealing with fermions, $M_i = m_i^2$ when dealing with bosons), ordered by their subscript $M_1 < M_2 < \cdots < M_N$, $\langle \cdot | \cdot \rangle$ is a pseudo-scalar product, and D a differential operator symmetric under $\langle \cdot | \cdot \rangle$, that is

$$\langle \phi_1 | D\phi_2 \rangle = \langle D\phi_1 | \phi_2 \rangle.$$

Generically, in the fermionic case D will be a first-order differential operator such as $D = i\partial$ and in the bosonic case a second-order differential operator such as $D = \delta d$ for differential forms or $D = \square(\frac{1}{2}P^{(2)} - P^{(s)})$ for HD-gravity [16].

Under this hypothesis, and through the lines presented in section 4, it is a trivial task to define a correspondence between $S_{HD}[\phi]$ and the action

$$\sum_{a=1}^N \prod_{b \neq a}^N (M_b - M_a) S_a[\phi_a] \quad (C2)$$

where

$$S_a[\phi_a] := \langle \phi_a | (D + M_a) \phi_a \rangle.$$

This can be done by following the same steps as in the spinorial case. First, define a Legendre transformation of the form (14)–(15) and then a linear redefinition in the form of (16)–(19) to select the propagating DOF. Formally, the same formulae are valid in the general case with the obvious identifications $\psi \leftrightarrow \phi$, $\bar{\psi} \leftrightarrow \langle \phi |$ and $m_l \leftrightarrow -M_l$.

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